Jeans solutions for triaxial galaxies

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Abstract. The Jeans equations relate the second-order velocity moments to the density and potential of a stellar system. For general three-dimensional stellar systems, there are three equations, but these are not very helpful, as they contain six independent moments. By assuming that the potential is triaxial and of separable Stäckel form, the mixed moments vanish in confocal ellipsoidal coordinates. The three Jeans equations and three remaining non-vanishing moments form a closed system of three highly-symmetric coupled first-order partial differential equations in three variables. They were first derived by Lynden–Bell in 1960, but have resisted solution by standard methods. Here we present the general solution by superposition of singular solutions.

1 Introduction

Much has been learned about the mass distribution and internal dynamics of galaxies by modeling their observed kinematics with solutions of the Jeans equations (e.g. [4]). The Jeans equations connect the second-order velocity moments (or the velocity dispersions, if the mean streaming motion is known) directly to the density and the gravitational potential of the galaxy, without the need to know the phase-space distribution function f. In nearly all cases there are fewer Jeans equations than velocity moments, so that additional assumptions have to be made about the degree of anisotropy. Furthermore, the resulting second moments may not correspond to a physical distribution function $f \geq 0$. These significant drawbacks have not prevented wide application of the Jeans approach to the kinematics of spherical and axisymmetric galaxies. Many (components of) galaxies have triaxial shapes ([2], [3]), including early-type bulges, bars, and giant elliptical galaxies. In this geometry, there are three Jeans equations, but little use has been made of them, as they contain six independent second moments, three of which have to be chosen ad-hoc (see e.g. [11]).

An exception is provided by the special set of triaxial mass models that have a gravitational potential of Stäckel form. In these systems, the Hamilton–Jacobi equation separates in confocal ellipsoidal coordinates ([19]), so that all orbits have three exact integrals of motion, which are quadratic in the velocities. The three mixed second-order velocity moments vanish, so that the three Jeans equations for the three remaining second moments form a closed system. Lynden–Bell ([13]) was the first to derive the explicit form of these Jeans equations. He showed that they constitute a highly symmetric set of three first-order partial differential equations for three unknowns, each of which is a function of the ellipsoidal coordinates, but he did not derive solutions.

When it was realized that the orbital structure in the triaxial Stäckel models is very similar to that in numerical models for triaxial galaxies with cores ([6], [16]), interest in the second moments increased, and the Jeans equations were solved for a number of special cases. These include the axisymmetric limits and elliptic discs ([8], [10]), triaxial galaxies with only thin tube orbits ([12]), and the scale-free limit ([11]). In all these cases the equations simplify to a two-dimensional problem, which can be solved with standard techniques after transforming two first-order equations into a single second-order equation in one dependent variable. However, these techniques do not carry over to a single third-order equation in one dependent variable, which is the best that one could expect to have in the general case. As a result, the latter has remained unsolved.

We have solved the two-dimensional case with an alternative solution method, which does not use the standard approach, but instead uses superposition of singular solutions. This approach can be extended to three dimensions, and provides the general solution for the triaxial case in closed form. We present the detailed solution method elsewhere ([23]), and here we summarise the main results. In ongoing work we will apply our solutions, and will use them together with the mean streaming motions ([20]) to study the properties of the observed velocity and dispersion fields of triaxial galaxies.

2 The Jeans equations for separable models

We define confocal ellipsoidal coordinates (λ, μ, ν) as the three roots for τ of

$$\frac{x^2}{\tau+\alpha} + \frac{y^2}{\tau+\beta} + \frac{z^2}{\tau+\gamma} = 1 \, , \tag{1}$$

with (x, y, z) the usual Cartesian coordinates, and with constants α, β and γ such that $-\gamma \leq \nu \leq -\beta \leq \mu \leq -\alpha \leq \lambda$. Surfaces of constant λ are ellipsoids, and surfaces of constant μ and ν are hyperboloids of one and two sheets, respectively. The confocal ellipsoidal coordinates are approximately Cartesian near the origin and become conical at large radii, i.e., equivalent to spherical coordinates.

We consider models with a gravitational potential of Stäckel form

$$V_S(\lambda, \mu, \nu) = -\frac{F(\lambda)}{(\lambda-\mu)(\lambda-\nu)} - \frac{F(\mu)}{(\mu-\nu)(\mu-\lambda)} - \frac{F(\nu)}{(\nu-\lambda)(\nu-\mu)} \, , \tag{2}$$

where $F(\tau)$ is an arbitrary smooth function. This potential is the most general form for which the Hamilton–Jacobi equation separates ([15], [18]) All orbits have three exact isolating integrals of motion, which are quadratic in the velocities (e.g. [6]). There are no irregular orbits, so that Jeans' theorem is strictly valid ([14]), and the distribution function f is a function of the three integrals. Therefore, out of the six symmetric second-order velocity moments, defined as

$$\langle v_i v_j \rangle(\boldsymbol{x}) = \frac{1}{\varrho} \iiint v_i v_j f(\boldsymbol{x}, \boldsymbol{v}) \, \mathrm{d}^3 v \, , \quad (i, j = 1, 2, 3), \tag{3}$$

with density ϱ, the three mixed moments vanish, and we are left with $\langle v_\lambda^2\rangle$, $\langle v_\mu^2\rangle$ and $\langle v_\nu^2\rangle$, related by three Jeans equations. These were first derived by Lynden–Bell ([13]), and can be written in the following form ([23])

$$\frac{\partial S_{\lambda\lambda}}{\partial\lambda} - \frac{S_{\mu\mu}}{2(\lambda-\mu)} - \frac{S_{\nu\nu}}{2(\lambda-\nu)} = g_1(\lambda,\mu,\nu) \ , \tag{4a}$$

$$\frac{\partial S_{\mu\mu}}{\partial\mu} - \frac{S_{\nu\nu}}{2(\mu-\nu)} - \frac{S_{\lambda\lambda}}{2(\mu-\lambda)} = g_2(\lambda,\mu,\nu) \ , \tag{4b}$$

$$\frac{\partial S_{\nu\nu}}{\partial\nu} - \frac{S_{\lambda\lambda}}{2(\nu-\lambda)} - \frac{S_{\mu\mu}}{2(\nu-\mu)} = g_3(\lambda,\mu,\nu) \ , \tag{4c}$$

where we have defined the diagonal components of the stress tensor

$$S_{\tau\tau}(\lambda,\mu,\nu) = \sqrt{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}\,\varrho\langle v_\tau^2\rangle \ , \qquad \tau=\lambda,\mu,\nu, \tag{5}$$

and the functions g_1, g_2 and g_3 depend on the density and potential (2) as

$$g_1(\lambda,\mu,\nu) = -\sqrt{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}\,\varrho\,\frac{\partial V_S}{\partial\lambda} \ , \tag{6}$$

where g_2 and g_3 follow from g_1 by cyclic permutation $\lambda\to\mu\to\nu\to\lambda$. Similarly, the three Jeans equations follow from each other by cyclic permutation. The stress components have to satisfy the following continuity conditions

$$S_{\lambda\lambda}(-\alpha,-\alpha,\nu) = S_{\mu\mu}(-\alpha,-\alpha,\nu) \ , \quad S_{\mu\mu}(\lambda,-\beta,-\beta) = S_{\nu\nu}(\lambda,-\beta,-\beta) \ , \tag{7}$$

at the focal ellipse ($\lambda=\mu=-\alpha$) and focal hyperbola ($\mu=\nu=-\beta$), respectively.

We prefer the form (5) for the stresses instead of the more common definition without the square root, since it results in more convenient and compact expressions. In self-consistent models, the density ϱ equals ϱ_S, with ϱ_S related to V_S by Poisson's equation. The Jeans equations, however, do not require self-consistency, so that we make no assumptions on the form of ϱ other than that it is triaxial, i.e., a function of (λ,μ,ν), and that it tends to zero at infinity.

3 The two-dimensional case

When two or all three of the constants α, β or γ in (1) are equal, the triaxial Stäckel models reduce to limiting cases with more symmetry and thus with fewer degrees of freedom. Solving the Jeans equations for oblate, prolate, elliptic disc and scale-free models reduces to the same two-dimensional problem ([10], [11], [23]), of which the simplest form is the pair of Jeans equations for Stäckel discs

$$\frac{\partial S_{\lambda\lambda}}{\partial\lambda} - \frac{S_{\mu\mu}}{2(\lambda-\mu)} = g_1(\lambda,\mu) \ , \tag{8a}$$

$$\frac{\partial S_{\mu\mu}}{\partial\mu} - \frac{S_{\lambda\lambda}}{2(\mu-\lambda)} = g_2(\lambda,\mu) \ , \tag{8b}$$

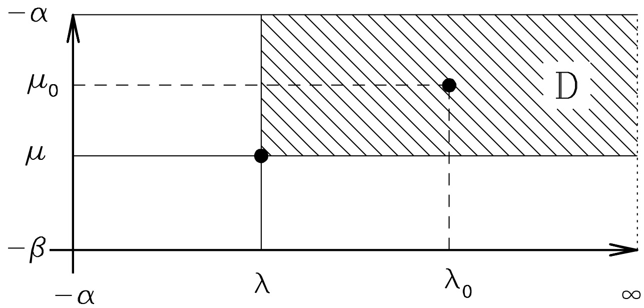

Fig. 1. *The* (λ_0, μ_0)*-plane. The total stress at a field point* (λ, μ) *consists of the weighted contributions from source points at* (λ_0, μ_0) *in the domain* D.

with at the foci ($\lambda = \mu = -\alpha$) the continuity condition

$$S_{\lambda\lambda}(-\alpha, -\alpha) = S_{\mu\mu}(-\alpha, -\alpha) \ . \tag{9}$$

In this case the stress components and the functions g_1 and g_2 are

$$S_{\tau\tau}(\lambda, \mu) = \sqrt{(\lambda - \mu)}\, \varrho \langle v_\tau^2 \rangle \quad (\tau = \lambda, \mu), \quad g_1(\lambda, \mu) = -\sqrt{(\lambda - \mu)}\, \varrho \frac{\partial V_S}{\partial \lambda} \ , \tag{10}$$

where g_2 follows from g_1 by interchanging $\lambda \leftrightarrow \mu$, and ϱ denotes a surface density.

The two Jeans equations (8) can be recast into a single second-order partial differential equation in either $S_{\lambda\lambda}$ or $S_{\mu\mu}$, which can be solved by employing standard techniques like Riemann's method ([5], [23]). However, these standard techniques do not carry over to the triaxial case, and we therefore introduce an alternative method, based on the superposition of singular solutions.

We consider a simpler form of (8) by substituting for g_1 and g_2, respectively $\tilde{g}_1 = 0$ and $\tilde{g}_2 = \delta(\lambda_0 - \lambda)\delta(\mu_0 - \mu)$. We refer to solutions of these simplified Jeans equations as *singular solutions*. Singular solutions can be interpreted as contributions to the stresses at a fixed field point (λ, μ) due to a source point in (λ_0, μ_0) (Fig. 1). The full stress at the field point can be obtained by adding all source point contributions, each with a weight that depends on the local density and potential. Once we know the singular solutions, we can use the superposition principle to construct the the solution of the full Jeans equations (8).

Since the derivative of a step-function $\mathcal{H}$ is equal to a delta-function, it follows that the singular solutions must have the form

$$\begin{aligned} S_{\lambda\lambda} &= A(\lambda, \mu)\mathcal{H}(\lambda_0 - \lambda)\mathcal{H}(\mu_0 - \mu) \ , \\ S_{\mu\mu} &= B(\lambda, \mu)\mathcal{H}(\lambda_0 - \lambda)\mathcal{H}(\mu_0 - \mu) - \delta(\lambda_0 - \lambda)\mathcal{H}(\mu_0 - \mu) \ . \end{aligned} \tag{11}$$

where the functions A and B must solve the homogeneous Jeans equations, i.e., (8) with zero right-hand side, and satisfy the following boundary conditions

$$A(\lambda_0, \mu) = \frac{1}{2(\lambda_0 - \mu)} \ , \quad B(\lambda, \mu_0) = 0 \ . \tag{12}$$

We solve this two-dimensional homogeneous boundary problem by superposition of particular solutions. We first derive a particular solution of the homogeneous Jeans equations with a free parameter z, which we assume to be complex. We then construct a linear combination of these particular solutions by integrating over z. We choose the integration contours in the complex z-plane, such that the boundary conditions (12) are satisfied simultaneously. The resulting homogeneous solutions are complex contour integrals, which can be evaluated in terms of the complete elliptic integral of the second kind, $E(w) \equiv \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \sqrt{1 - w \sin^2 \theta}$, and its derivative $E'(w)$, as

$$A = \frac{E(w)}{\pi(\lambda_0 - \mu)}, \quad B = -\frac{2wE'(w)}{\pi(\lambda_0 - \lambda)}, \quad \text{with} \quad w = \frac{(\lambda_0 - \lambda)(\mu_0 - \mu)}{(\lambda_0 - \mu_0)(\lambda - \mu)}. \tag{13}$$

We obtain a similar system of simplified Jeans equations by interchanging the expressions for $\tilde{g}_1$ and $\tilde{g}_2$. The singular solutions of this simplified system follow from (11) by interchanging $\lambda \leftrightarrow \mu$ and $\lambda_0 \leftrightarrow \mu_0$ at the same time.

To find the solution to the full Jeans equations (8) at (λ, μ), we multiply the latter singular solutions and (11) by $g_1(\lambda_0, \mu_0)$ and $g_2(\lambda_0, \mu_0)$ respectively, and integrate over $D = \{(\lambda_0, \mu_0)\colon \lambda \le \lambda_0 \le \infty, \mu \le \mu_0 \le -\alpha\}$ (Fig. 1). This gives the first two integrals of the two equations (14a) and (14b) below. The remaining terms are due to the non-vanishing stress at the boundary $\mu = -\alpha$, and are found by multiplying the singular solutions (11), evaluated at $\mu_0 = -\alpha$, by $-S_{\mu\mu}(\lambda_0, -\alpha)$ and integrating over λ_0 in D. The final result for the solution of the Jeans equations (8) for Stäckel discs, after using the evaluations (13), is

$$\begin{aligned} S_{\lambda\lambda}(\lambda, \mu) = & \int_\lambda^\infty \mathrm{d}\lambda_0 \int_\mu^{-\alpha} \mathrm{d}\mu_0 \left[-g_1(\lambda_0, \mu_0) \frac{2wE'(w)}{\pi(\mu_0 - \mu)} + g_2(\lambda_0, \mu_0) \frac{E(w)}{\pi(\lambda_0 - \mu)} \right] \\ & - \int_\lambda^\infty \mathrm{d}\lambda_0 \, g_1(\lambda_0, \mu) - \int_\lambda^\infty \mathrm{d}\lambda_0 \, S_{\mu\mu}(\lambda_0, -\alpha) \left[\frac{E(w)}{\pi(\lambda_0 - \mu)} \right]_{\mu_0 = -\alpha}, \end{aligned} \tag{14a}$$

$$\begin{aligned} S_{\mu\mu}(\lambda, \mu) = & \int_\lambda^\infty \mathrm{d}\lambda_0 \int_\mu^{-\alpha} \mathrm{d}\mu_0 \left[-g_1(\lambda_0, \mu_0) \frac{E(w)}{\pi(\lambda - \mu_0)} - g_2(\lambda_0, \mu_0) \frac{2wE'(w)}{\pi(\lambda_0 - \lambda)} \right] \\ & - \int_\mu^{-\alpha} \mathrm{d}\mu_0 \, g_2(\lambda, \mu_0) + S_{\mu\mu}(\lambda, -\alpha) - \int_\lambda^\infty \mathrm{d}\lambda_0 \, S_{\mu\mu}(\lambda_0, -\alpha) \left[-\frac{2wE'(w)}{\pi(\lambda_0 - \lambda)} \right]_{\mu_0 = -\alpha}. \end{aligned} \tag{14b}$$

The solution depends on ϱ and V_S through g_1 and g_2. This means that, for given ϱ and V_S, the solution is uniquely determined once we have prescribed $S_{\mu\mu}$ at the boundary $\mu = -\alpha$. At this boundary, $S_{\lambda\lambda}$ is related to $S_{\mu\mu}$ by the first Jeans equation (8a), evaluated at $\mu = -\alpha$, up to an integration constant, which is fixed by the continuity condition (9). We are thus free to specify either of the two stress components at $\mu = -\alpha$.

4 The general case

The singular solution method introduced in the previous section can be extended to three dimensions to solve the Jeans equations (4) for triaxial Stäckel models. Although the calculations are more complex for a triaxial model, the stepwise solution method is similar to that in two dimensions.

We simplify the Jeans equations (4) by setting two of the three functions g_1, g_2 and g_3 to zero and the remaining equal to $\delta(\lambda_0-\lambda)\delta(\mu_0-\mu)\delta(\nu_0-\nu)$. In this way, we obtain three similar simplified systems ($i=1,2,3$), each with three singular solutions $S_i^{\tau\tau}(\lambda,\mu,\nu;\lambda_0,\mu_0,\nu_0)$ ($\tau=\lambda,\mu,\nu$), that describe the stress components at a fixed field point (λ,μ,ν) due to a source point in (λ_0,μ_0,ν_0).

The singular solutions have a form that is similar to that in the two-dimensional case (11). They consist of combinations of step-functions and delta-functions multiplied by functions that are the solutions of homogeneous boundary problems. The functions that must solve a two-dimensional homogeneous boundary problem can be found as in §3, and can be expressed in terms of complete elliptic integrals, cf. (13). The singular solutions in the general case also contain three functions A, B and C that must solve the triaxial homogeneous Jeans equations, i.e., (4) with zero right-hand side, and satisfy three boundary conditions. This three-dimensional homogeneous boundary problem can be solved by integrating a *two*-parameter particular solution over both its complex parameters, and choosing the combination of contours such that the three boundary conditions are satisfied simultaneously. The resulting homogeneous solutions A, B and C are products of complex contour integrals, and can be evaluated as sums of products of complete *hyper*elliptic integrals.

To find the solution of the full Jeans equations (4) , we multiply each singular solution $S_i^{\tau\tau}$ by $g_i(\lambda_0,\mu_0,\nu_0)$, so that the contribution from the source point naturally depends on the local density and potential. Then, for each coordinate $\tau=\lambda,\mu,\nu$, we add the three weighted singular solutions, and integrate over a finite volume within the physical region $-\gamma\le\nu\le-\beta\le\mu\le-\alpha\le\lambda$. This results in the following general solution of the Jeans equations (4) for triaxial Stäckel models

$$
\begin{aligned}
S_{\tau\tau}(\lambda,\mu,\nu) = {}& \int_\lambda^{\lambda_e}\mathrm{d}\lambda_0 \int_\mu^{\mu_e}\mathrm{d}\mu_0 \int_\nu^{\nu_e}\mathrm{d}\nu_0 \sum_{i=1}^{3} g_i(\lambda_0,\mu_0,\nu_0)\, S_i^{\tau\tau}(\lambda,\mu,\nu;\lambda_0,\mu_0,\nu_0) \\
& - \int_\mu^{\mu_e}\mathrm{d}\mu_0 \int_\nu^{\nu_e}\mathrm{d}\nu_0\, S_{\lambda\lambda}(\lambda_e,\mu_0,\nu_0)\, S_1^{\tau\tau}(\lambda,\mu,\nu;\lambda_e,\mu_0,\nu_0) \\
& - \int_\nu^{\nu_e}\mathrm{d}\nu_0 \int_\lambda^{\lambda_e}\mathrm{d}\lambda_0\, S_{\mu\mu}(\lambda_0,\mu_e,\nu_0)\, S_2^{\tau\tau}(\lambda,\mu,\nu;\lambda_0,\mu_e,\nu_0) \\
& - \int_\lambda^{\lambda_e}\mathrm{d}\lambda_0 \int_\mu^{\mu_e}\mathrm{d}\mu_0\, S_{\nu\nu}(\lambda_0,\mu_0,\nu_e)\, S_3^{\tau\tau}(\lambda,\mu,\nu;\lambda_0,\mu_0,\nu_e)\,, \qquad (15)
\end{aligned}
$$

with $\tau = \lambda, \mu, \nu$. Whereas the integration limits λ, μ and ν are fixed due to the position of the field point, the limits λ_e, μ_e and ν_e are not, and may be any value in the corresponding physical ranges, i.e., $\lambda_e \in [-\alpha, \infty]$, $\mu_e \in [-\beta, -\alpha]$ and $\nu_e \in [-\gamma, -\beta]$, but $\lambda_e \neq -\alpha$. The latter choice would lead to solutions which generally have the incorrect radial fall-off, and hence are non-physical. If we choose $\lambda_e = \infty$, there is no contribution from the second line in (15) due to vanishing stress at large distance. If we furthermore take $\mu_e = -\alpha$ and $\nu_e = -\beta$, the integration volume becomes the three-dimensional extension of D (Fig. 1).

Whereas the volume integral in (15) already solves the inhomogeneous Jeans equations (4) , the three area integrals are needed to obtain the correct values at the boundary surfaces $\lambda = \lambda_e$, $\mu = \mu_e$ and $\nu = \nu_e$. On each of these surfaces the three stress components are related by two of the three Jeans equations (4) and the continuity conditions (7). Since the (weight) functions g_i are known for given ϱ and V_S, this means that the solution (15) yields all three stresses everywhere in the triaxial model, once one of the stress components is prescribed on the three boundary surfaces. If we take $\lambda_e = \infty$ and $\mu_e = \nu_e = -\beta$, the contributing boundary surfaces reduce to the single (x, z)-plane, containing the long and the short axis of the galaxy. This compares well with Schwarzschild ([17]), who used the same plane to start his numerically calculated orbits from.

5 Discussion and conclusions

Eddington ([9]) showed that the velocity ellipsoid in a triaxial galaxy with a separable potential of Stäckel form is everywhere aligned with the confocal ellipsoidal coordinate system in which the equations of motion separate. Lynden–Bell ([13]) derived the three Jeans equations which relate the three principal stresses to the potential and the density. Solutions were found for the various two-dimensional limiting cases, but with methods that do not carry over to the general case, which remained unsolved. We have presented an alternative solution method, based on the superposition of singular solutions (see [23] for details). This approach, unlike the standard techniques, can be generalised to solve the three-dimensional system. The resulting solutions contain complete (hyper)elliptic integrals, which can be evaluated in a straightforward way.

The general Jeans solution is not unique, but requires specification of principal stresses at certain boundary surfaces, given a separable triaxial potential and a triaxial density distribution (not necessarily the one that generates the potential). These boundary surfaces can be taken to be the plane containing the long and the short axis of the galaxy, and, more specifically, the part that is crossed by all three families of tube orbits and the box orbits.

The set of all Jeans solutions (15) contains all the stresses that are associated with the physical distribution functions $f \geq 0$, but, as in the case of spherical and axisymmetric models, also contains solutions which are unphysical, e.g., those associated with distribution functions that are negative in some parts of phase space. The many examples of the use of spherical and axisymmetric Jeans models in the literature suggest nevertheless that the Jeans solutions can be of significant use.

While triaxial models with a separable potential do not provide an adequate description of the nuclei of galaxies with cusped luminosity profiles and a massive central black hole ([7]), they do catch much of the orbital structure at larger radii, and in some cases even provide a good approximation of the galaxy potential. The solutions for the mean streaming motions, i.e., the first velocity moments of the distribution function, are helpful in understanding the variety of observed velocity fields in giant elliptical galaxies and constraining their intrinsic shapes (e.g. [1], [21], [22]). We expect that the projected velocity dispersion fields that can be derived from our Jeans solutions will be similarly useful, and, in particular, that they can be used to establish which combinations of viewing directions and intrinsic axis ratios are firmly ruled out by the observations.

It is remarkable that the entire Jeans solution can be written down by means of classical methods. This suggests that similar solutions can be found for the higher dimensional analogues of (4) , most likely involving hyperelliptic integrals of higher order. It is also likely that the higher-order velocity moments for the separable triaxial models can be found by similar analytic means, but the effort required may become prohibitive.

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